

THE MODULE STRUCTURE OF THE EQUIVARIANT K -THEORY OF THE BASED LOOP GROUP OF $SU(2)$

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ABSTRACT. Let $G = SU(2)$ and let ΩG denote the space of based loops in $SU(2)$. We explicitly compute the $R(G)$ -module structure of the topological equivariant K -theory $K_G^*(\Omega G)$ and in particular show that it is a direct product of copies of $K_G^*(\text{pt}) \cong R(G)$. (We intend to describe in detail the $R(G)$ -algebra (i.e. product) structure of $K_G^*(\Omega G)$ in a forthcoming companion paper.) Our proof uses the geometric methods for analyzing loop spaces introduced by Pressley and Segal (and further developed by Mitchell). However, Pressley and Segal do not explicitly compute equivariant K -theory and we also need further analysis of the spaces involved since we work in the equivariant setting. With this in mind, we have taken this opportunity to expand on the original exposition of Pressley-Segal in the hope that in doing so, both our results and theirs would be made accessible to a wider audience.

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1. INTRODUCTION

Let G be a compact connected Lie group. The G -equivariant topological K -theory $K_G^*(X)$ of a topological G -space X is an object of intrinsic interest, carrying information about X which reflects the G -action on X . The space G itself, with G acting by conjugation, and its space of (continuous) based loops ΩG with the induced (pointwise) action, are two examples of natural and important G -spaces. For Lie groups G , the ordinary and Borel-equivariant cohomology rings $H^*(G)$, $H^*(\Omega G)$, $H_G^*(G)$, and $H_G^*(\Omega G)$ were computed decades ago (with contributions from many people), and these results are by now considered classical; the same is true of the computations of the ordinary K -theory rings $K^*(G)$ and $K^*(\Omega G)$. However, computing equivariant K -theory of these spaces proved to be more difficult. For instance, $K_G^*(G)$ was only recently computed by Brylinski and Zhang in 2000 [6].

For the remainder of this paper we mainly restrict attention to the special case $G = SU(2)$. The chief contribution of this manuscript is a concrete computation of the module structure of $K_G^*(\Omega G)$ for the specific case $G = SU(2)$. Here we view $K_G^*(\Omega G)$ as a module over $K_G^* = K_G^*(\text{pt}) \cong R(G)$. For the following let $\Omega_{\text{poly}} G$ denote the subspace of *polynomial loops* in G , and $\Omega_{\text{poly},r} G$ the subspace of polynomial loops of degree $\leq r$. (See equation (2.1).) The spaces $\Omega_{\text{poly},r} G$ form a filtration of $\Omega_{\text{poly}} G$. With this notation in place we may state the main theorem of this manuscript (Theorem 5.5), which

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asserts in particular that $K_G^*(\Omega G)$ (respectively $K_T^*(\Omega G)$) is an inverse limit of free $R(G)$ -modules (respectively $R(T)$ -modules).

Theorem 1.1. *Let $G = SU(2)$ and let T denote its maximal torus. Let ΩG denote the space of based loops in G , equipped with the pointwise conjugation action of G . The $R(G)$ -module (respectively $R(T)$ -module) $K_G^*(\Omega G)$ (respectively $K_T^*(\Omega G)$) can be described as follows:*

$$\begin{aligned} K_G^q(\Omega G) &\cong K_G^q(\Omega_{\text{poly}} G) \cong \varprojlim K_G^q(\Omega_{\text{poly},r} G) \cong \begin{cases} \prod_{r=0}^{\infty} R(G) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd} \end{cases} \\ K_T^q(\Omega G) &\cong K_T^q(\Omega_{\text{poly}} G) \cong \varprojlim K_T^q(\Omega_{\text{poly},r} G) \cong \begin{cases} \prod_{r=0}^{\infty} R(T) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd} \end{cases} \end{aligned}$$

□

This theorem should not be surprising to experts for two reasons. Firstly, the computation for the non-equivariant case, using an analogous filtration, follows from the work of many other authors: for instance, James [14] described in 1955 a filtration of spaces of the form $\Omega \Sigma X$, which applies to our situation of $G = SU(2)$ since $SU(2) \cong S^3 \cong \Sigma S^2$, while Pressley and Segal develop a theory for general loop groups ΩG in [22], which was further developed by Mitchell in [21]). Indeed, the technical geometric tools for our argument are G -equivariant analogues of the ideas of Pressley and Segal. However, our geometric results are not immediate corollaries of those in [22], mainly due to Theorem 4.7. The non-equivariant analogue of Theorem 4.7 in [22] is a description of a certain space as a product of contractible spaces [22, (8.4.4)], while in Theorem 4.7, we instead get a non-trivial bundle over \mathbb{P}^1 . This geometric distinction is relevant in our analysis. Secondly, statements similar to Theorem 1.1 for the $(T \times S^1)$ -equivariant K -theory $K_{T \times S^1}^*(\Omega G)$ can be deduced by Kac-Moody methods (see Kostant-Kumar [16]) or GKM methods (see e.g. Harada-Henriques-Holm [8]). However, our G -equivariant result is not an immediate corollary of these torus-equivariant results since, for instance, it is not always the case for a G -space X that

$$K_G^*(X) \cong K_T^*(X)^W$$

(cf. for example [11, Example 4.8]). For this reason we worked instead with G -equivariant analogues of the approach in [22]. (In fact, as it turns out, direct computation confirms the isomorphism $K_G^*(\Omega G) \cong K_T^*(\Omega G)^W$ is satisfied in our case.)

We now summarize the strategy of our computation in some more detail. Let $\Omega_{\text{poly}} G$ and $\Omega_{\text{psm}} G$ denote the subspaces of polynomial and piecewise smooth loops, respectively, in ΩG . (Both are defined more precisely below.) One of our key steps is to prove that there are G -equivariant homotopy equivalences

$$(1.1) \quad \Omega_{\text{poly}} G \simeq_G \Omega_{\text{psm}} G \simeq_G \Omega G.$$

This reduces our computation to that of $K_G^*(\Omega_{\text{poly}} G)$. Our second essential strategy is to analyze the G -filtration of $\Omega_{\text{poly}} G$ by the spaces $\Omega_{\text{poly},r} G$ for $r \in \mathbb{Z}_{>0}$, consisting of loops of polynomial degree $\leq r$. More specifically, we prove that the filtration quotients

$$(1.2) \quad \Omega_{\text{poly},r} G / \Omega_{\text{poly},r-1} G$$

are G -homeomorphic to Thom spaces of complex G -vector bundles over \mathbb{P}^1 , implying that its equivariant K -theory can be computed via the (equivariant) Thom isomorphism theorem. From this, a computation of $K_G(\Omega_{\text{poly}} G)$ is obtained by induction and taking the inverse limit. In order to achieve the results mentioned above, we introduce and use (following [22]) the Grassmannian $\text{Gr}^z(\mathcal{K})$, where \mathcal{K} is a separable Hilbert space described precisely below. One of the reasons the space $\text{Gr}^z(\mathcal{K})$ is useful is because there is a subspace $\text{Gr}_{\text{bdd},r}^z(\mathcal{K}) \subset \text{Gr}^z(\mathcal{K})$ which is G -equivariantly homeomorphic to $\Omega_{\text{poly},r} G$. Thus our proofs proceed by analyzing appropriate subspaces of $\text{Gr}^z(\mathcal{K})$, instead of working directly with ΩG .

As already mentioned, the broad outline of our analysis follows the well-known work of Pressley and Segal [22]. As such, in the current manuscript we have taken this opportunity to significantly expand on the exposition in [22]; in doing so, we hope that both our results and those in [22] will be made accessible to a broader audience.

Notation. We standardize some notation and collect well-known facts to be used throughout.

- The Lie group G is always $SU(2)$ unless otherwise noted.
- T is the maximal torus of G given by $\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in S^1 \right\}$.
- $W \cong S_2$ is the Weyl group of G .
- $R(T)$ is the representation ring of T and similarly $R(G)$ is the representation ring of G .
- $K_T(\text{pt}) \cong R(T)$ and $K_G(\text{pt}) \cong R(G)$.
- Complex projective space \mathbb{P}^1 can be G -equivariantly identified with G/T , where G acts on G/T by the usual translation.

2. THE GRASSMANNIAN $\text{Gr}^z(\mathcal{K})$ AND ITS SUBSPACES

In this section we define a separable Hilbert space \mathcal{K} and its associated Grassmannian $\text{Gr}^z(\mathcal{K})$. Our discussion follows [22]. The main results are Theorems 2.5 and 2.9, which provide G -equivariant identifications of appropriate subspaces of $\text{Gr}^z(\mathcal{K})$ with the G -spaces

$$\Omega_{\text{psm}}U(n), \Omega_{\text{psm}}SU(n), \Omega_{\text{poly},r}U(n),$$

and $\Omega_{\text{poly},r}SU(n)$, respectively. These identifications allow us, in later sections, to use the language of Grassmannians in order to prove results about $\Omega_{\text{psm}}G$ and $\Omega_{\text{poly}}G$. In this section only, our discussion is valid for $U(n)$ and $SU(n)$ for any $n \geq 2$.

First we quickly recall the definitions of the spaces of (based) loops in question. Let H be any Lie group. As is standard, we let ΩH denote the space of continuous based loops in H with basepoint the identity in H . We define the **piecewise smooth (based) loops** from S^1 to H to be

$$\Omega_{\text{psm}}H := \{f \in \Omega H \mid f \text{ is piecewise smooth}\}.$$

Evidently, $\Omega_{\text{psm}}H \subseteq \Omega H$. Now consider the special case $H = U(n)$. Following [22], we also define the space of **polynomial based loops** $\Omega_{\text{poly}}U(n)$ as the set of maps $S^1 \rightarrow U(n)$ which can be expressed as Laurent polynomials in z , where z is the parameter on the circle S^1 . More precisely, for $r \geq 0$ we define

$$(2.1) \quad \Omega_{\text{poly},r}U(n) := \left\{ f : S^1 \rightarrow U(n) \mid f(1) = \mathbf{1}_{n \times n}, f = \sum_{j=-r(n-1)}^r a_j z^j, a_j \in M(n \times n, \mathbb{C}) \right\},$$

where $\mathbf{1}_{n \times n}$ denotes the identity matrix. Here the a_j are constant $n \times n$ complex matrices, and $f(z)$ is required to be unitary (in particular invertible) for all $z \in S^1$. An element in $\Omega_{\text{poly},r}U(n)$ may also be viewed as an element of $\Omega_{\text{poly},r'}U(n)$ for any $r' > r$. Via these natural inclusions we may define

$$\Omega_{\text{poly}}U(n) := \bigcup_{r=0}^{\infty} \Omega_{\text{poly},r}U(n)$$

We refer to $\Omega_{\text{poly}}U(n)$ as the **(space of) polynomial (based) loops in $U(n)$** .

Now let $\mathcal{H} = L^2(S^1)$ and set $\mathcal{K} := \mathcal{H} \otimes \mathbb{C}^n$. Let z denote the parameter on $S^1 \subseteq \mathbb{C}$. If we normalize the measure on S^1 so that $\mu(S^1) = 1$, then $\{z^\ell \mid \ell \in \mathbb{Z}\}$ forms an orthonormal Hilbert space basis for \mathcal{H} . Define

$$\mathcal{H}_+ := \text{closed subspace of } \mathcal{H} \text{ spanned by } \{z^\ell \mid \ell \geq 0\}$$

and

$$\mathcal{H}_- := \mathcal{H} \ominus \mathcal{H}_+ = \text{closed subspace of } \mathcal{H} \text{ spanned by } \{z^\ell \mid \ell < 0\}.$$

Let $\mathcal{K}_+ := \mathcal{H}_+ \otimes \mathbb{C}^n$ and $\mathcal{K}_- := \mathcal{H}_- \otimes \mathbb{C}^n$. We now define the Grassmannian (also called the affine Grassmannian) associated to \mathcal{K} as

$$(2.2) \quad \text{Gr}^z(\mathcal{K}) := \{\text{closed subspaces } W \text{ of } \mathcal{K} \mid zW \subset W\}.$$

For $r \geq 0$, we define the following important subspaces of $\text{Gr}^z(\mathcal{K})$:

$$(2.3) \quad \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) := \{W \in \text{Gr}^z(\mathcal{K}) \mid z^r \mathcal{K}_+ \subset W \subset z^{-r(n-1)} \mathcal{K}_+\}.$$

We also define the subspaces of **bounded weight** by

$$(2.4) \quad \text{Gr}_{\text{bdd}}^z(\mathcal{K}) := \bigcup_r \text{Gr}_{\text{bdd},r}^z(\mathcal{K}).$$

For $f \in \Omega \text{GL}(n)$, let $M_f : \mathcal{K} \rightarrow \mathcal{K}$ denote the multiplication operator $(M_f(h))(z) := f(z)h(z)$ where the right hand side is the usual multiplication of the vector $h(z) \in \mathbb{C}^n$ by the matrix $f(z) \in \text{GL}(n)$. We denote by

$$W_f := \overline{M_f(\mathcal{K}_+)} \subset \mathcal{K}$$

the closure of the image of \mathcal{K}_+ under M_f . Note that since $f(z)$ is continuous and invertible for z in the compact set S^1 , the operator M_f on \mathcal{K} is both bounded and invertible. Moreover, since multiplication by $f(z)$ and z commute and because \mathcal{K}_+ is closed under multiplication by z , we have $W_f \in \text{Gr}^z(\mathcal{K})$. Thus the map

$$(2.5) \quad \alpha : \Omega \text{GL}(n) \rightarrow \text{Gr}^z(\mathcal{K}), \quad f \mapsto W_f$$

is well-defined.

The group $SU(n)$ acts on $\Omega \text{GL}(n)$ by pointwise conjugation and on $\mathcal{K} := \mathcal{H} \otimes \mathbb{C}^n$ (and hence also on \mathcal{K}_+) by acting on the second factor. Since the action of $SU(n)$ and multiplication by z commute on \mathcal{K} , there is also an induced $SU(n)$ -action on $\text{Gr}^z(\mathcal{K})$. We have the following.

Lemma 2.1. *The map α in (2.5) is $SU(n)$ -equivariant.*

Proof. For $g \in SU(n)$,

$$\alpha(g \cdot f(\cdot)) = M_{g \cdot f(\cdot)}(\mathcal{K}_+) = M_{gf(\cdot)g^{-1}}(\mathcal{K}_+) = \{gf(\cdot)g^{-1}h(\cdot) \mid h \in \mathcal{K}_+\}.$$

Since $SU(n)$ acts on $\mathcal{K}_+ = \mathcal{H}_+ \otimes \mathbb{C}^n$ through its second factor only, it follows that $g\mathcal{K}_+ = \mathcal{K}_+$, and $h(z) \in \mathcal{K}_+$ if and only if $gh(z) \in \mathcal{K}_+$ for any $g \in SU(n)$. Therefore

$$\{gf(z)g^{-1}h(z) \mid h(z) \in \mathcal{K}_+\} = \{gf(z)h(z) \mid h(z) \in \mathcal{K}_+\} = gM_f(\mathcal{K}_+) = g\alpha(f(z))$$

as desired. \square

Now let

$$(2.6) \quad \alpha_{\text{psm}} := \alpha|_{\Omega_{\text{psm}}U(n)} : \Omega_{\text{psm}}U(n) \rightarrow \text{Gr}^z(\mathcal{K})$$

denote the restriction of α to $\Omega_{\text{psm}}U(n)$. We also define

$$(2.7) \quad \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) := \alpha(\Omega_{\text{psm}}U(n)) \subset \text{Gr}^z(\mathcal{K})$$

to be the image under α of the piecewise smooth loops in $U(n)$, i.e., the image of α_{psm} .

Our next major goal, recorded in Theorem 2.5, is to show that the restriction of α to the piecewise smooth loops $\Omega_{\text{psm}}U(n)$ is in fact an equivariant homeomorphism onto its image. We accomplish this by defining a map $\beta : \alpha(\Omega_{\text{psm}}\text{GL}(n)) \rightarrow \Omega_{\text{psm}}U(n)$ which we show to be an equivariant retraction to α_{psm} . The construction and argument requires several preliminary steps.

For a closed subspace W of \mathcal{K} , let $P_+^W : W \rightarrow \mathcal{K}_+$ and $P_-^W : W \rightarrow \mathcal{K}_-$ denote the orthogonal projections onto \mathcal{K}_+ and \mathcal{K}_- respectively. Given a bounded linear operator $B \in B(\mathcal{K})$, the inclusions $\mathcal{K}_{\pm} \rightarrow \mathcal{K}$ and projections $\mathcal{K} \rightarrow \mathcal{K}_{\pm}$ give a description of B as a matrix of operators

$$B = \begin{pmatrix} B_{++} & B_{+-} \\ B_{-+} & B_{--} \end{pmatrix}$$

as in [22, page 80]. In particular $(M_f)_{++}$ is the composite $\mathcal{K}_+ \xrightarrow{(M_f)|_{\mathcal{K}_+}} W_f \xrightarrow{P_+^{W_f}} \mathcal{K}_+$. Finally, for a Fredholm operator F , let $\text{Ind}(F) = \dim \text{Ker}(F) - \dim \text{Coker}(F)$ denote the index of F . More generally, we define the index of W by $\text{Ind}(W) = \dim(\text{Ker}(P_+^W)) - \dim(\text{Coker}(P_+^W))$ for a closed subspace W of \mathcal{K} for which both are finite-dimensional. In the case of a subspace W_f arising from a function f whose associated $(M_f)_{++}$ is Fredholm, we have $\text{Ind}(W_f) = \text{Ind}((M_f)_{++})$ since $(M_f)|_{\mathcal{K}_+} : \mathcal{K}_+ \rightarrow W_f$ is an injection.

We may now state and prove the following.

Lemma 2.2. *Suppose $f \in \Omega_{\text{psm}} \text{GL}(n)$. Then $(M_f)_{++}$ is a Fredholm operator and its index $\text{Ind}((M_f)_{++})$ equals $-n$ times the degree of the homotopy class of the function $z \mapsto \det(f(z))$ in $\pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$.*

Proof. The fact that $(M_f)_{++}$ is a Fredholm operator is an immediate corollary of [22, Proposition 6.3.1]. Although their statement is for the continuously differentiable case, in fact their proof does not require more than piecewise continuous differentiability.

Since the integers are discrete, it follows that $\text{Ind}((M_f)_{++})$ depends only on the homotopy class of $\det f(z)$. Therefore to verify the formula for the index it suffices to consider the special case where $f(z) = \begin{pmatrix} z^k & 0 \\ 0 & 1 \end{pmatrix}$ for some integer k . If $k \geq 0$ we get $\dim \text{Ker}(M_f)_{++} = 0$ and $\dim \text{Coker}(M_f)_{++} = nk$ and if $k \leq 0$ we get $\dim \text{Ker}(M_f)_{++} = nk$ and $\dim \text{Coker}(M_f)_{++} = 0$ and so the formula holds in both cases. \square

Motivated by the above lemma, we wish to focus attention on the subset of $\text{Gr}_{\alpha(\text{psm})}(\mathcal{K})$ with associated index 0. Specifically, we define the **special Grassmannian** by

$$(2.8) \quad S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) := \{W \in \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) \mid \text{Ind}(W) = 0\}.$$

We also define

$$(2.9) \quad S \text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K}) := \{W \in S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) \mid \dim \text{Ker}(P_+^W) = \dim \text{Coker}(P_+^W) \leq r\}$$

and

$$S \text{Gr}_{\alpha(\text{psm}),=r}^z(\mathcal{K}) := \{W \in S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) \mid \dim \text{Ker}(P_+^W) = \dim \text{Coker}(P_+^W) = r\}.$$

Lemma 2.6 below explains this terminology: namely, if f takes values in the special linear group $SL(n)$, then its image under α is in the special Grassmannian.

Similarly we define

$$(2.10) \quad S \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) := \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) \cap S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) = \{W \in \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) \mid \text{Ind}(W) = 0\}.$$

This is precisely the subset of the Grassmannian which we can equivariantly identify with the polynomial loops $\Omega_{\text{poly},r} G$ of degree $\leq r$ mentioned in the introduction, cf. Theorem 2.9 below.

With the above results in place we now give an explicit construction of a map β which we will show is an equivariant inverse to α_{psm} . Suppose $W \in \alpha(\Omega_{\text{psm}} \text{GL}(n))$, so $W = W_f = M_f(\mathcal{K}_+)$ for some $f \in \Omega_{\text{psm}} \text{GL}(n)$. By Lemma 2.2 we know that $(M_f)_{++}$ is Fredholm. Using this fact, Pressley and Segal show in [22, p.126] that $\dim(W \ominus zW) = n$. Choose an ordered orthonormal basis

$$B = (w_1(z), w_2(z), \dots, w_n(z))$$

for $W \ominus zW$. Let $N_B(z)$ be the $n \times n$ matrix whose j th column is formed from the components of $w_j \in \mathcal{H} \otimes \mathbb{C}^n$ with respect to the standard basis $\{e_1, e_2, \dots, e_n\}$ for the \mathbb{C}^n in the second factor. It is shown in [22, p.126] that $N_B(z) \in U(n)$ for each $z \in S^1$. Thus $z \mapsto N_B(1)^{-1} N_B(z)$ is a well-defined loop in $\Omega U(n)$. We now define the map β as

$$(2.11) \quad \beta : \alpha(\Omega_{\text{psm}} \text{GL}(n)) \rightarrow \Omega_{\text{psm}} U(n), \quad \beta(W)(z) := N_B(1)^{-1} N_B(z).$$

We must first prove the following lemma.

Lemma 2.3. *The map β in (2.11) is well-defined.*

Proof. We first show that the image of β , which a priori is an element of $\Omega U(n)$, in fact lands in $\Omega_{\text{psm}} U(n)$. This follows from the construction of N_B and the fact that f is by definition piecewise smooth. Next note that for a different choice of ordered orthonormal basis $B' = (w'_1, w'_2, \dots, w'_n)$ of $W \ominus zW$, the matrix $N_{B'}(z)$ would be related to N_B by $N_{B'}(z) = AN_B(z)$ where $A \in U(n)$ is the (constant) linear transformation taking the ordered basis B to B' . Therefore

$$N_{B'}(1)^{-1}N_{B'}(z) = N_B(1)^{-1}A^{-1}AN_B(z) = N_B(1)^{-1}N_B(z).$$

Hence β is well-defined. \square

We next prove that β respects the relevant group actions.

Lemma 2.4. *The map β in (2.11) is $U(n)$ -equivariant.*

Proof. Let $W \in \alpha(\Omega_{\text{psm}} GL(n))$ with choice of ordered basis $B = (w_1(z), w_2(z), \dots, w_n(z))$ for $W \ominus zW$. Then $gB(z) := (gw_1(z), gw_2(z), \dots, gw_n(z))$ is a valid ordered basis for $g \cdot W \ominus z(g \cdot W)$. Therefore $N_{gB}(z) = N_B(z)g^{-1}$ and so

$$\beta(g \cdot W) = gN_B(0)^{-1}N_B(z)g^{-1} = g \cdot \beta(W)$$

as desired. \square

We are ready to prove that α_{psm} is an equivariant homeomorphism onto its image, with equivariant inverse given by the above map β . This is an equivariant analogue of [22, Theorem 8.3.2].

Theorem 2.5.

The map α_{psm} is an $SU(n)$ -equivariant homeomorphism from $\Omega_{\text{psm}} U(n)$ to its image $\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$, with (equivariant) inverse given by $\beta|_{\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})}$.

Proof. For $f \in \Omega_{\text{psm}} U(n)$, the set $B = \{f(z)e_1, \dots, f(z)e_n\}$ forms an orthonormal basis for $W_f \ominus zW_f$. Writing the components of these vectors as columns of a matrix simply reproduces the matrix $f(z)$. That is, $N_B(z) = f(z)$ and so $\beta(\alpha(f)) = f$. Now suppose $W \in \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. The definition of β requires that the rows of $\beta(W)$ form an orthonormal basis for $W \ominus zW$, which means that they form a generating set for W as a $\mathbb{C}[z]$ -module. Hence $\alpha(\beta(W)) = W$. Since α and β are continuous by [22, page 129] and are $SU(n)$ -equivariant, it follows that α_{psm} is an $SU(n)$ -equivariant homeomorphism from $\Omega_{\text{psm}} U(n)$ to its image. \square

Lemma 2.6. *If $f \in \Omega_{\text{psm}} SL(n)$ then $W_f \in S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$.*

Proof. If $f \in \Omega_{\text{psm}} GL(n)$ then $W_f = W_{\tilde{f}}$ where $\tilde{f} := \beta \circ \alpha(f) \in \Omega_{\text{psm}} U(n)$. This exhibits W_f as an element of $\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. If $f \in \Omega_{\text{psm}} SL(n)$ then $z \mapsto \det(f(z))$ is the constant function 1. This has degree 0, so by Lemma 2.2 we conclude $\text{Ind}((M_f)_{++}) = 0$ and hence $W_f \in S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. \square

We next show that α and β also behave well with respect to the filtrations on the spaces $\Omega_{\text{poly}} U(n)$ and $\text{Gr}_{\text{bdd}}^z(\mathcal{K})$. We first record a simple lemma used in the proof.

Lemma 2.7. *Let $p(z)$ be a polynomial with complex coefficients which has a nonzero constant term and satisfies $|p(z)| = 1$ for all $z \in S^1$. Then $p(z)$ is a constant.*

Proof. Let $m = \deg p$. Set $f(z) := p(z)\bar{p}(z^{-1})$ where $\bar{p}(w) = \overline{p(w)}$ is the polynomial obtained by taking the complex conjugate of each of the coefficients in $p(z)$. Then $f(z)$ is analytic on the complement of $\{0\}$, with a pole of order m at 0. On the unit circle we have

$$f(z) = p(z)\bar{p}(z^{-1}) = p(z)\bar{p}(\bar{z}) = |p(z)|^2 = 1.$$

If two analytic functions agree on a convergent sequence then they are equal and so $f(z) = 1$ on $\mathbb{C} \setminus \{0\}$. But then the singularity of $f(z)$ at the origin is removable, which implies that $m = 0$. \square

Proposition 2.8. *Let α and β be the maps defined in (2.5) and (2.11) respectively. Then:*

$$(1) \quad \alpha(\Omega_{\text{poly}, r} U(n)) \subset \text{Gr}_{\text{bdd}, r}^z(\mathcal{K}).$$

- (2) The restriction of α to $\Omega_{\text{poly},r}U(n)$ is a surjection to $\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$. In particular, $\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$ is a subset of $\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. Thus β is defined on $\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$ and $\beta(\text{Gr}_{\text{bdd},r}^z(\mathcal{K})) \subset \Omega_{\text{poly},r}SU(n)$.
- (3) $\beta(S \text{Gr}_{\text{bdd},r}^z(\mathcal{K})) \subset \Omega_{\text{poly},r}SU(n)$.

Proof.

- (1) Let $f \in \Omega_{\text{poly},r}U(n)$. Then $z^{r(n-1)}f(z)$ is a polynomial in z and so if $h(z) \in \mathcal{K}_+$ then $z^{r(n-1)}f(z)h(z) \in \mathcal{K}_+$. It follows that $f(z)h(z) \in z^{-r(n-1)}\mathcal{K}_+$. Thus $W_f \subset z^{-r(n-1)}\mathcal{K}_+$. Next we show $z^r\mathcal{K}_+ \subset W_f$. Let $h(z) \in z^r\mathcal{K}_+$. Then $z^{-r}h(z) \in \mathcal{K}_+$. Since $f(z) \in \Omega_{\text{poly},r}U(n)$, it is a Laurent polynomial with powers of z ranging between $z^{-r(n-1)}$ and z^r . Hence its adjoint $f(z)^* = f(z)^{-1}$ has powers of z between z^{-r} and $z^{r(n-1)}$. In particular $z^r f(z)^{-1}$ is polynomial (i.e. has no negative powers of z). Hence

$$f(z)^{-1}(h(z)) = z^r f(z)^{-1}(z^{-r}h(z))$$

lies in \mathcal{K}_+ , which in turn implies $h(z) = f(z)(f(z)^{-1}(h(z)))$ lies in $f(z)(\mathcal{K}_+) \subset W_f$, as desired.

- (2) Given part (1), the equality $\alpha(\Omega_{\text{poly},r}U(n)) = \text{Gr}_{\text{bdd},r}^z(\mathcal{K})$ can be verified by counting dimensions. Alternatively we can construct the preimage (under α) of a subspace $W \in \text{Gr}_{\text{bdd},r}^z(\mathcal{K})$ as follows. The usual Gram-Schmidt procedure in the finite dimensional vector space

$$W/z^r\mathcal{K} \subset z^{-r}\mathcal{K}/z^r\mathcal{K} \cong \mathbb{C}^{nr}$$

can be used to construct an orthonormal basis for $W \ominus zW$ in which the components of all elements are Laurent polynomials with nonzero coefficients of z^k only for $-r(n-1) \leq k \leq r$. (Note that the normalization portion of this Gram-Schmidt process requires only divisions by positive real numbers, not polynomials, so the resulting elements are still (Laurent) polynomials.) Now apply the construction given in the definition of β . The resulting function will lie in $\Omega_{\text{poly},r}U(n)$.

- (3) Suppose $W \in S \text{Gr}_{\text{bdd}}^z$. By part (2), $\beta(W) \in \Omega U(n)$ and we must show that

$$\det(\beta(W))(z) = 1$$

for all $z \in S^1$. Let $h(z) = \det(\beta(W))(z)$. Using $h(z) \in U(n)$, we know $h(1) = 1$ and $|h(z)| = 1$ for all $z \in S^1$. Also $\text{Ind}(W) = 0$, and so $h(z) \simeq 1$. Since $h(z)$ is a Laurent polynomial, there exists r such that $p(z) := z^r h(z)$ is a polynomial with nonzero constant term. The polynomial $p(z)$ satisfies $|p(z)| = 1$ for all $z \in S^1$, so by Lemma 2.7, $p(z)$ is a constant function. We can evaluate the constant using $h(1) = 1$ to deduce that $p(z) \equiv 1$. Thus $h(z) = z^{-r}$. The fact that $h(z) \simeq 1$ tells us that $r = 0$, so $h(z) \equiv 1$.

□

Now let

$$(2.12) \quad \alpha_{\text{poly},r} := \alpha|_{\Omega_{\text{poly},r}U(n)} : \Omega_{\text{poly},r}U(n) \rightarrow \text{Gr}_{\text{bdd},r}^z(\mathcal{K})$$

denote the restriction of α_{psm} to $\Omega_{\text{poly},r}U(n)$. We are ready to state and prove the analogues of Theorem 2.5 for the relevant subspaces of $\Omega_{\text{psm}}U(n)$. The first claim of the theorem below is an analogue of [22, Proposition 8.3.3(i)].

Theorem 2.9.

- (1) The map $\alpha_{\text{poly},r}$ is an $SU(n)$ -equivariant homeomorphism from $\Omega_{\text{poly},r}U(n)$ to $\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$.
- (2) The restriction of α_{psm} to $\Omega_{\text{psm}}SU(n)$ is an $SU(n)$ -equivariant homeomorphism from $\Omega_{\text{psm}}SU(n)$ to its image in $S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) := \{W \in \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) \mid \text{Ind}(W) = 0\}$.
- (3) The restriction of $\alpha_{\text{poly},r}$ to $\Omega_{\text{poly},r}SU(n)$ is an $SU(n)$ -equivariant homeomorphism from

$$\Omega_{\text{poly},r}SU(n)$$

$$\text{to } S \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) := \{W \in \text{Gr}_{\text{bdd},r}^z(\mathcal{K}) \mid \text{Ind}(W) = 0\}.$$

Proof. The first claim follows from Theorem 2.5 and Proposition 2.8, since the restriction of a homeomorphism to a subspace induces a homeomorphism onto its image. Restricting to the connected component of the identity in $\Omega_{\text{psm}}U(n)$ and $\Omega_{\text{poly},r}U(n)$ respectively gives the last two claims. \square

3. DESCRIPTION OF FILTRATION QUOTIENTS AS THOM SPACES

Henceforth we restrict attention to the case $n = 2$. In particular, we return to our main case $G = SU(2)$. Our main result in the previous section, Theorem 2.9, shows that the spaces $\Omega_{\text{poly},r}G$, which provide a natural filtration of $\Omega_{\text{poly}}G$, may be (equivariantly) identified with $S\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$, so these spaces will be the main focus of our analysis below. For simplicity we introduce the notation

$$(3.1) \quad F_{2r} := S\text{Gr}_{\text{bdd},r}^z(\mathcal{K}).$$

The main result of this brief section is a concrete geometric description of the quotients F_{2r}/F_{2r-2} as Thom spaces of vector bundles. Here and below, γ denotes the tautological bundle over \mathbb{P}^1 . Also we equip \mathbb{C}^2 with the standard hermitian metric, and let \perp denote the orthogonal complement with respect to this metric. The following definition is useful for our description of F_{2r}/F_{2r-2} .

Definition 3.1. Let τ denote the G -equivariant complex line bundle over \mathbb{P}^1 whose total space is

$$\{(u, v) \mid u \in S^3 \subset \mathbb{C}^2, v \in (u^\perp)\} / \sim$$

where the equivalence relation is given by $(u, v) \sim (\zeta u, \zeta v)$ for $\zeta \in S^1$, and with projection to \mathbb{P}^1 given by $[(u, v)] \mapsto [u] \in \mathbb{P}^1$. The G -action is defined by $g \cdot [(u, v)] := [(gu, gv)]$.

The notation τ is justified by the following proposition.

Proposition 3.2. *The bundle τ of Definition 3.1 is G -equivariantly isomorphic to the tangent bundle of \mathbb{P}^1 .*

Proof. The tangent bundle of \mathbb{P}^1 can be identified with $\text{Hom}(\gamma, \gamma^\perp)$, as Milnor shows in the proof of [20, Theorem 14.10]. (Although [20] discusses only the non-equivariant case, it is in fact easy to check that the maps defined there are G -equivariant.) Thus it suffices to show that τ is G -equivariantly isomorphic to $\text{Hom}(\gamma, \gamma^\perp)$. An element $[(u, v)]$ in the total space of τ uniquely specifies a linear map $\phi_{[(u, v)]} : \gamma \rightarrow \gamma^\perp$ by setting $\phi_{[(u, v)]}(u) = v$. By linearity, $\phi_{[(u, v)]}(\zeta u) = \zeta v$, so this is well-defined on equivalence classes, and it is straightforward to see this is a bijective correspondence which is equivariant and linear on fibers. \square

We now proceed to the main result of this section, Proposition 3.4.

Lemma 3.3. *Let $r \in \mathbb{Z}$ with $r > 0$. Let $W \in F_{2r} \setminus F_{2r-2}$. Then there exists $w \in W$ of the form*

$$w = z^{-r}u_0 + z^{-r+1}u_1 + \dots + z^{r-2}u_{2r-2} + z^{r-1}u_{2r-1}$$

with $u_j \in \mathbb{C}^2$, $u_0 \neq 0$ and $u_j \perp u_0$ for $j > 0$. Moreover, up to a nonzero complex scalar multiple, w is uniquely determined by W .

Proof. That there exists such a w follows from the assumption that W is in F_{2r} but not in F_{2r-2} . The uniqueness of such w up to multiplication by a scalar multiple follows from the assumption that W is closed under multiplication by z and the fact that $\dim_{\mathbb{C}}(W/z^r\mathcal{K}_+) = 2r$ (which in turn follows from the assumption that $\text{Ind}((M_f)_{++}) = 0$). \square

Our main geometric proposition follows immediately from the preceding discussion.

Proposition 3.4. *Let $r \in \mathbb{Z}$ and $r \geq 0$. The quotient space F_{2r}/F_{2r-2} is G -equivariantly homeomorphic to $\text{Thom}(\tau^{2r-1})$.* \square

Using Proposition 3.4 and the Thom isomorphism in equivariant K -theory ([3, Theorem 6.1.4] or [7, Theorem 3.1]) yields the following.

Theorem 3.5.

$$K_G^q(F_{2r}) \cong \begin{cases} \prod_{k=0}^r R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Considering F_{2r} as a T -space under the natural restriction of the G -action to its maximal torus T , we also have

$$K_T^q(F_{2r}) \cong \begin{cases} \prod_{k=0}^r R(T) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Proof. In the exact sequence

$$(3.2) \quad \dots \rightarrow K_G^*(F_{2r}, F_{2(r-1)}) \rightarrow K_G^*(F_{2r}) \rightarrow K_G^*(F_{2(r-1)}) \rightarrow \dots$$

associated to the pair $(F_{2r}, F_{2(r-1)})$, the equivariant Thom isomorphism gives

$$K_G^*(F_{2r}, F_{2(r-1)}) \cong \tilde{K}_G^q(\text{Thom}(\tau^{2r-1})) \cong K_G^q(\mathbb{P}^1) \cong \begin{cases} R(G) \oplus R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Thus the exact sequence decomposes into a collection of short exact sequences. These short exact sequences split since $K_G^*(F_{2(r-1)})$ is a free $R(G)$ -module by induction (where the base case for the induction is $F_0 = \text{pt}$). The first statement then follows. The proof of the statement for K_T^* is identical. \square

Our final result for $K_G(\Omega_{\text{poly}}G)$ is obtained by taking an inverse limit.

Theorem 3.6. *Let $G = SU(2)$ and let T denote its maximal torus. Let $\Omega_{\text{poly}}G$ denote the space of based polynomial loops in G , equipped with the pointwise conjugation action of G . The $R(G)$ -module (respectively $R(T)$ -module) $K_G^*(\Omega_{\text{poly}}G)$ (respectively $K_T^*(\Omega_{\text{poly}}G)$) can be described as follows:*

$$\begin{aligned} K_G^q(\Omega_{\text{poly}}G) &\cong \varprojlim K_G^q(\Omega_{\text{poly},r}G) \cong \begin{cases} \prod_{r=0}^{\infty} R(G) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd;} \end{cases} \\ K_T^q(\Omega_{\text{poly}}G) &\cong \varprojlim K_T^q(\Omega_{\text{poly},r}G) \cong \begin{cases} \prod_{r=0}^{\infty} R(T) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases} \end{aligned}$$

Proof. Since

$$\bigcup_r F_{2r} \cong_G \Omega_{\text{poly}}(SU(2))$$

by Theorem 2.9, the computation of $K_G(\Omega_{\text{poly}}SU(2))$ as an $R(G)$ -module is obtained by taking the inverse limit of $K_G^*(F_{2r})$. More specifically, the Milnor exact sequence [24] implies that $K_G^*(\Omega_{\text{poly}}G)$ is given by $\varprojlim K_G^*(\Omega_{\text{poly},r}G)$ or equivalently by $\varprojlim K_G^*(F_{2r})$. The result follows. \square

4. THE G -HOMOTOPY EQUIVALENCE $S\text{Gr}_{\text{bdd},r}^{\prime z}(\mathcal{K}) \rightarrow S\text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K})$

The goal of the rest of the manuscript is to prove that the inclusion $\Omega_{\text{poly}}G \hookrightarrow \Omega G$ is a G -homotopy equivalence; we do this by proving separately that the inclusion maps $\Omega_{\text{poly}}G \hookrightarrow \Omega_{\text{psm}}G$ and $\Omega_{\text{psm}}G \hookrightarrow \Omega G$ are both G -homotopy equivalences. This then reduces the computation of $K_G^*(\Omega G)$ to that of $K_G^*(\Omega_{\text{poly}}G)$, which was recorded in Theorem 3.6 above. To this end, the goal of the present section is to show that the inclusion $S\text{Gr}_{\text{bdd},r}^{\prime z}(\mathcal{K}) \hookrightarrow S\text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K})$ is a G -equivariant homotopy equivalence for any fixed $r > 0$ (Theorem 4.20), where $S\text{Gr}_{\text{bdd},r}^{\prime z}(\mathcal{K})$ is a certain subspace (similar in spirit to $S\text{Gr}_{\text{bdd},r}^z(\mathcal{K})$) to be defined precisely below. This is the main technical ingredient in our proof that $\Omega_{\text{poly}}G \hookrightarrow \Omega_{\text{psm}}G$ is a G -homotopy equivalence (Theorem 5.3).

We begin with an explicit description of a map

$$(4.1) \quad \pi : S\text{Gr}_{\alpha(\text{psm}),=r}^z(\mathcal{K}) \rightarrow \mathbb{P}^1.$$

The map π will play a significant role in the technical arguments below, where we show (Proposition 4.5) that π is a G -homotopy equivalence between a certain subset Σ_r^G of $S \operatorname{Gr}_{\alpha(\operatorname{psm}),=r}^z(\mathcal{K})$ and \mathbb{P}^1 , which in turn allows us to prove our main geometric result (Theorem 4.7).

We will formulate the construction of π in an entirely coordinate-free manner, in particular without choosing either a maximal torus of G or an ordered basis of \mathbb{C}^2 . Suppose

$$W = W_f = \alpha(f) \in S \operatorname{Gr}_{\alpha(\operatorname{psm})}^z(\mathcal{K})$$

for some $f \in \Omega_{\operatorname{psm}} G$. By the earlier discussion, $(M_f)_{++}$ is a Fredholm operator with index 0, so we define

$$r = r(W) := \dim \operatorname{Ker}((M_f)_{++}) = \dim \operatorname{Coker}((M_f)_{++}).$$

We sometimes refer to $r(W)$ as the *rank* of W .

Given a Laurent series $g(z) = \sum_k a_k z^k$ with $a_k \in \mathbb{C}^2$ we let $\deg(g)$ denote the maximum of the set $\{k \mid a_k \neq 0\}$ or ∞ if there is no maximum. Thus $\deg(g) \in \mathbb{Z} \cup \{\infty\}$. Note that if $\deg(g) < 0$ then there is no ‘Taylor part’ to the Laurent series, i.e. there are no non-zero terms $a_k z^k$ with $k \geq 0$. Also for any $k \in \mathbb{Z}$ we denote by $\langle z^k \rangle \otimes \mathbb{C}^2$ the 2-dimensional subspace of $\mathcal{H} \otimes \mathbb{C}^2$ spanned by the vectors with a z^k coefficient.

Suppose now $W = W_f$ where $r(W) = r > 0$. We may think of $w \in W$ as a Laurent series in the variable z with coefficients in \mathbb{C}^2 . Consider the set

$$S_W := \{w \in W \mid \deg(w) = -1\}.$$

Observe that $S_W \neq \emptyset$ since otherwise $\operatorname{Ker}((M_f)_{++}) = 0$. Let V be the subspace of $\langle z^{-1} \rangle \otimes \mathbb{C}^2 \cong \mathbb{C}^2$ spanned by the leading coefficients of the elements of S_W .

Lemma 4.1. *Let $W = W_f$ with $r(W) > 0$, and let V be as above. Then $\dim_{\mathbb{C}}(V) = 1$.*

Proof. Since $S_W \neq \emptyset$, we know $\dim_{\mathbb{C}}(V) > 0$. On the other hand, if $\dim_{\mathbb{C}}(V) = 2$ then these elements span $\langle z^{-1} \rangle \otimes \mathbb{C}^2$ and then the leading coefficients of the set $z^{k+1} S_W$ would span $\langle z^k \rangle \otimes \mathbb{C}^2$ for all $k \geq 0$, which would in turn imply $\operatorname{Coker}((M_f)_{++}) = 0$. This is a contradiction since we assumed $r = r(W) > 0$. The conclusion follows. \square

From Lemma 4.1, for each $W = W_f$ with $r(W) > 0$ we obtain a well-defined 1-dimensional subspace V of \mathbb{C}^2 . Hence our concrete description of the map π of (4.1) is given by

$$(4.2) \quad \pi(W) = V \in \mathbb{P}^1$$

where we view the one-dimensional subspace V of \mathbb{C}^2 as an element in \mathbb{P}^1 as usual.

We now use the map π to define a homomorphism $\lambda_W : S^1 \rightarrow SU(2)$ associated to W . First consider the case $r = r(W) > 0$. Let $v \in S^3 \subset \mathbb{C}^2$ be a representative for $\pi(W) = V$ and choose $u \in S^3$ such that $u \perp v$. The corresponding homomorphism $\lambda_W : S^1 \rightarrow SU(2)$ is defined by $(\lambda_W(z))(u) = z^{r(W)}u$ and $(\lambda_W(z))(v) = z^{-r(W)}v$. More concretely, when written in the u, v -basis we have

$$(4.3) \quad \lambda_W(z) = \begin{pmatrix} z^{r(W)} & 0 \\ 0 & z^{-r(W)} \end{pmatrix}.$$

(Note the elements u and v are determined by W only up to multiplication by an element of S^1 , but the resulting homomorphism λ_W is independent of these choices.) In the case $r(W) = 0$ we simply define $\lambda_W(z) \equiv 1$ to be the trivial homomorphism taking every element to the identity in $SU(2)$.

More detailed information about subspaces of rank r is given in the following proposition.

Proposition 4.2. *Let $W = W_f$ with $r(W_f) = r > 0$. Then*

- (1) *We have $r = -\min\{k \mid W_f \text{ has an element of degree } k\}$.*
- (2) *A basis for the kernel of the orthogonal projection $W_f \rightarrow \mathcal{K}_+$ is given by the set*

$$\{x, zx, z^2x, \dots, z^{r-1}x\}$$

where $x \in W_f$ satisfies $\deg(x) = -r$. The subspace V of Lemma 4.1 is spanned by $z^{r-1}x$.

(3) *The orthogonal projection from W_f to $\lambda_W(\mathcal{K}_+)$ is an isomorphism.*

Proof. Recall that $a(z) \mapsto f(z)a(z)$ gives an isomorphism $\mathcal{K}_+ \rightarrow W_f$ and in particular it gives an isomorphism from $\text{Ker}(M_f)_{++}$ to the kernel K of the orthogonal projection $W_f \rightarrow \mathcal{K}_+$.

Since

$$\dim K = \dim \text{Ker}(M_f)_{++} = r > 0,$$

there must exist elements in W_f having negative degree (zero Taylor part). If $\deg(y) = -m < 0$ then $\{y, zy, z^2y, \dots, z^{m-1}y\}$ are linearly independent elements of K , so $m \leq r$. Thus the set

$$(4.4) \quad \min\{k \mid W \text{ has an element of degree } k\}$$

is bounded below. Now let x be an element of W with degree equal to the minimum of (4.4) and let $m := -\deg(x)$. Then $0 < m \leq r$. Since $\deg(x)$ is minimum, $x \neq zy$ for any $y \in W$. Consider the set

$$B := \{x, zx, z^2x, \dots, z^{m-1}x\} \in K.$$

We claim that B forms a basis for the kernel K . Suppose for a contradiction there exists $w \in K$ which is not in the linear span of B . By multiplication by powers of z , we may assume without loss of generality that $\deg(w) = -1$. Using that $x \neq zy$ for any y , we see that $\{z^{m-1}x, w\}$ are linearly independent in $V := \langle z^{-1} \rangle$, contradicting Lemma 4.1. Therefore there is no such w , and so B is a basis for K . In particular,

$$r = m = -\min\{k \mid W \text{ has an element of degree } k\}.$$

This establishes the first two parts of the proposition.

Part (2) tells us that W contains no elements of negative degree outside of the linear span of B , and since $\dim \text{Coker}((M_f)_{++}) = r$, it follows that the least degree of any element of W_f outside of the closed $\mathbb{C}[z]$ module generated by x is r . In other words, there exists y with $\deg(y) = r$ such that $W = W_{x,y}$ and so orthogonal projection W_f to $\lambda_W(\mathcal{K}_+)$ is an isomorphism. \square

Proposition 4.3. *Suppose $W \in F_{2r} := S \text{Gr}_{\text{bdd}, r}^z(\mathcal{K})$. Then $r(W) \leq r$ and $r(W) = r$ if and only if $W \in F_{2r} \setminus F_{2r-2}$.*

Proof. $W \in F_{2k} \setminus F_{2k-2}$ for some $k \leq r$. By Lemma 3.3 there exists $w \in W$ such that

$$w = z^{-k}u_0 + \dots + z^{k-1}u_{2k-1}.$$

Set $u = u_0$ and choose $v \perp u$.

Let x be the element of least degree in W . By inspection of the form of W , $x = z^jw + cz^kv$ for some $c \in \mathbb{C}$ and some j . Hence $-\deg(x) \leq k \leq r$. But then by Proposition 4.2, $-\deg(x) = r(W)$. \square

We also need the following notation. For λ a homomorphism $\lambda : S^1 \rightarrow SU(2)$, we also view λ as an element of $\Omega SU(2)$. We let \mathcal{K}_λ denote the subspace $\lambda(\mathcal{K}_+)$. Let \mathcal{O} denote the ring of infinite series $a(z) = \sum_{n=0}^{\infty} a_n z^n$ in non-negative powers of z which converge on the closed unit disk D^2 in \mathbb{C} . (In particular, by assumption such $a(z)$ are holomorphic on the interior of the unit disk.) By slight abuse of notation we sometimes view an element $a(z)$ in \mathcal{O} as a function on the boundary S^1 , while at other times we view it as a function on D^2 .

Following [22] we also introduce the following sets of matrix-valued functions. First let

$$\mathcal{N}^- := \left\{ \begin{pmatrix} 1 + z^{-1}a(z^{-1}) & b(z^{-1}) \\ z^{-1}c(z^{-1}) & 1 + z^{-1}d(z^{-1}) \end{pmatrix} \mid a(z), b(z), c(z), d(z) \in \mathcal{O} \right\}$$

be the set of 2×2 matrix-valued functions $A(z)$, where the matrix entries are of the above form (and in particular are holomorphic on the region $\{\|z\| > 1\}$) and such that $A(\infty)$ is upper-triangular with 1's on the diagonal. Restricting this set slightly further we also define

$$N^- := \{A(z) \in \mathcal{N}^- \mid A(z) \text{ is invertible for all } z \text{ with } \|z\| \geq 1\}$$

and, restricting still further, we set

$$L_1^- := \left\{ A(z) \in N^- \mid A(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The definition of L_1^- in particular implies that elements in L_1^- have the form

$$\begin{pmatrix} 1 + z^{-1}a(z^{-1}) & z^{-1}b(z^{-1}) \\ z^{-1}c(z^{-1}) & 1 + z^{-1}d(z^{-1}) \end{pmatrix}$$

i.e. the constant term in the upper-right corner must be equal to 0.

Extending the notation of Section 2 slightly, for $A(z) : S^1 \rightarrow GL(2, \mathbb{C})$ any polynomial loop, *not* necessarily based at the identity, we denote by $M_A : \mathcal{K} \rightarrow \mathcal{K}$ the multiplication operator defined by $M_A(h)(z) := A(z) \cdot h(z)$ and let

$$W_A := \overline{M_A(\mathcal{K}_+)}$$

denote the closed subspace of \mathcal{K} which is the closure of the image of \mathcal{K}_+ . More concretely, if we let $u = u(z) : S^1 \rightarrow \mathbb{C}^2$ and $v = v(z) : S^1 \rightarrow \mathbb{C}^2$ denote the first and second columns of A respectively, then W_A is the closure of the span of the elements in $\mathcal{K} := \mathcal{H} \otimes \mathbb{C}^2$ of the form

$$\{z^k u(z), z^k v(z) \mid k \geq 0\}.$$

Motivated by this, given 2 vector-valued functions $u(z), v(z) : S^1 \rightarrow \mathbb{C}^2$ which are everywhere linearly independent, we also denote

$$W_{u,v} := W_A$$

where the matrix $A := [u \ v]$ is obtained by putting $u(z)$ in the left column and $v(z)$ in the right column.

For any homomorphism $\lambda : S^1 \rightarrow SU(2)$ there exists an orthonormal basis u_λ, v_λ of \mathbb{C}^2 with respect to which $\lambda(z)$ is diagonal with $\lambda(z) = \text{diag}(z^r, z^{-r})$ for some $r \geq 0$. The integer r uniquely determined by λ and for $r > 0$ the basis $\{u_\lambda, v_\lambda\}$ is uniquely determined up to common scalar multiple.

Multiplying the matrices gives

$$\lambda L_1^- \lambda^{-1} = \left\{ \begin{pmatrix} 1 + z^{-1}a(z^{-1}) & z^{2r-1}b(z^{-1}) \\ z^{-2r-1}c(z^{-1}) & 1 + z^{-1}d(z^{-1}) \end{pmatrix} \mid \text{invertible for } \|z\| \geq 1 \text{ and } a(w), b(w), c(w), d(w) \in \mathcal{O} \right\}.$$

Following Pressley and Segal, [22, 8.6.3(iv)], we now define

$$(4.5) \quad U_\lambda := \lambda L_1^- \lambda^{-1} \mathcal{K}_\lambda = \{W_A \mid A(z) \in \lambda L_1^-\}$$

where here we view a 2×2 matrix as a linear transformation on \mathbb{C}^2 written with respect to the basis $\{u_\lambda, v_\lambda\}$, and W_A denotes the closed subspace $M_A(\mathcal{K}_+)$ defined above. More concretely, U_λ consists of closed subspaces $W_{u,v}$ in \mathcal{K} where $u = u(z), v = v(z)$ are of the form

$$u(z) = \begin{pmatrix} z^r(1 + z^{-1}a(z^{-1})) \\ z^{-r-1}c(z^{-1}) \end{pmatrix}, \quad v(z) = \begin{pmatrix} z^{r-1}b(z^{-1}) \\ z^{-r}(1 + z^{-1}d(z^{-1})) \end{pmatrix},$$

where both u and v are written with respect to the basis u_λ, v_λ , and $a(z), b(z), c(z), d(z) \in \mathcal{O}$, and

$$\begin{pmatrix} 1 + z^{-1}a(z^{-1}) & z^{2r-1}b(z^{-1}) \\ z^{-2r-1}c(z^{-1}) & 1 + z^{-1}d(z^{-1}) \end{pmatrix}$$

is invertible for z with $\|z\| \geq 1$. (We will give an alternative, and more conceptual, description of U_λ below.)

We will also need to analyze the following subset of U_λ . Namely, we define

$$(4.6) \quad \Sigma_\lambda := \left\{ W_{u,v} \in U_\lambda \mid u(z) = \begin{pmatrix} z^r(1 + z^{-1}a(z^{-1})) \\ z^{-r-1}c(z^{-1}) \end{pmatrix} \text{ and } v(z) = \begin{pmatrix} z^{-r}b(z^{-1}) \\ z^{-r}(1 + z^{-1}d(z^{-1})) \end{pmatrix} \right\}.$$

In other words, if $r > 0$ then Σ_λ consists of those subspaces in U_λ which can be expressed as $W_{u,v}$ where the u_λ coordinate of v has no non-zero coefficients for $z^{-\ell}$ for $\ell < r$. Note that if $W \in \Sigma_\lambda$, then $r(W) = r$ since it can be seen from the definition to contain an element of degree $-r$ but none of lower degree.

For a homomorphism $\lambda : S^1 \rightarrow SU(2)$, we let $|\lambda| \geq 0$ denote the unique non-negative integer such that $\lambda(z) = \text{diag}(z^{|\lambda|}, z^{-|\lambda|})$ with respect to some (orthonormal) basis. For a fixed integer $r \geq 0$ we now define

$$\Sigma_r^G := \bigcup_{|\lambda|=r} \Sigma_\lambda,$$

i.e. Σ_r^G is the G -orbit of Σ_λ . Similarly let

$$U_r^G := \bigcup_{|\lambda|=r} U_\lambda.$$

These spaces play the roles analogous to that of Σ_λ and U_λ , respectively, in the arguments of Pressley-Segal.

Remark 4.4. (This is a technical remark for readers intending to work explicitly with these spaces U_r^G .)

If $W \in U_r^G$ then r is not uniquely determined by W . Indeed, let e, f be the standard basis for \mathbb{C}^2 , $r = 2$, and consider $W = W_{z^2e, ze+z^{-2}f}$. Then the only λ with $|\lambda| = 2$ for which $W \notin U_\lambda$ is $\lambda e = z^{-2}e$, $\lambda f = z^2f$, corresponding to the ordered orthonormal basis f, e . However we can also express this same subspace as $W = W_{ze+z^{-2}f, z^{-1}f}$, which exhibits W as an element of U_1^G . As we shall see later, $r(W)$ is the least r such that $W \in U_r^G$.

We now proceed to an analysis of the topology of U_r^G and Σ_r^G . We first show that Σ_r^G is G -homotopy equivalent to \mathbb{P}^1 . We then show that U_r^G can be regarded as the total space of a rank $2r - 1$ complex vector bundle over Σ_r^G . In fact we are able to identify the bundle explicitly as the pullback $\pi^*(\tau^{2r-1})$, where π is the map to \mathbb{P}^1 defined above and τ is the tangent bundle to \mathbb{P}^1 . Our main geometric statement is Theorem 4.7, which leads to the homotopy equivalence $S\text{Gr}_{\text{bdd}, r}^{\prime z} \hookrightarrow S\text{Gr}_{\alpha(\text{psm}), r}^z$ of Theorem 4.20 and ultimately to the homotopy equivalence of Theorem 5.3.

Fix $r > 0$. For $x \in \mathbb{P}^1$, choose a unit vector $v \in \mathbb{C}^2$ representing the line x , and also choose a unit vector u such that u, v form the left and right columns respectively of an element of $SU(2)$. Define $s_r(x) \in \Sigma_r^G$ by

$$s_r(x) := W_{z^r u, z^{-r} v}.$$

This is well-defined since the subspace $W_{z^r u, z^{-r} v}$ is independent of the choices made for u and v . It is straightforward to check that $s_r : \mathbb{P}^1 \rightarrow \Sigma_r^G$ is G -equivariant, and also that $\pi \circ s_r = 1_{\mathbb{P}^1}$.

Notice that L_1^- is contractible, with an explicit contraction given by $H_t(A)(z) := A(t^{-1}z)$. This leads to the following proposition, which is the G -equivariant analogue of the fact, recorded in [22, Theorem 8.6.3], that Σ_λ is contractible in the non-equivariant setting.

Proposition 4.5. *The map $\pi : \Sigma_r^G \rightarrow \mathbb{P}^1$ is a G -homotopy equivalence for all $r > 0$, with G -homotopy inverse s_r .*

Proof. It is straightforward from its definition that π is G -equivariant. The G -homotopy coming from $H_t(A)(z) := A(t^{-1}z)$ is given explicitly as follows. Given an element $W_{u,v} \in \Sigma_\lambda$ for $u(z), v(z)$ of the form given in (4.6), we can define

$$u_t = u(t, z) := \begin{pmatrix} z^r(1 + tz^{-1}a(tz^{-1})) \\ (tz^{-1})^{r+1}c(tz^{-1}) \end{pmatrix}, \quad v_t = v(t, z) := \begin{pmatrix} (tz^{-1})^r b(tz^{-1}) \\ z^{-r}(1 + tz^{-1}d(tz^{-1})) \end{pmatrix}$$

and consider the corresponding subspaces W_{u_t, v_t} . This evidently defines a G -equivariant deformation retraction taking $W_{u,v}$ to $s_r(\pi(W_{u,v}))$, as desired. \square

In the case $r = 0$, there is only one homomorphism $\lambda : S^1 \rightarrow SU(2)$ with $r(\lambda) = 0$. Therefore $\Sigma_0^G \cong \Sigma_\lambda$ where λ is the trivial homomorphism. Thus the next statement follows from the contraction $H_t(A)(z) := A(t^{-1}z)$ of L_1 in the same way.

Proposition 4.6. *The space $U_0^G = \Sigma_0^G$ is G -equivariantly contractible.*

The next theorem is our main technical geometric result. It identifies U_r^G as the total space of a complex vector bundle over Σ_r^G obtained by pullback via the G -homotopy equivalence $\pi : \Sigma_r^G \rightarrow \mathbb{P}^1$ discussed above. Recall that τ denotes the tangent bundle to \mathbb{P}^1 .

Theorem 4.7. *Let $r > 0$. Then the space U_r^G is G -homeomorphic to the total space of the bundle $\pi^*(\tau^{2r-1})$ over Σ_r^G .*

Proof. Following the notation from Section 3, the total space $E(\tau^{2r-1})$ of the bundle τ^{2r-1} over \mathbb{P}^1 can be described as

$$E(\tau^{2r-1}) = \{(v, x) \mid v \in S^3 \subset \mathbb{C}^2, x \in (v^\perp)^{2r-1}\} / \sim$$

where $(v, x) \sim (\zeta v, \zeta x)$ for $\zeta \in S^1$. Thus the total space of the pullback bundle $\pi^*(\tau^{2r-1})$ over Σ_r^G is

$$E(\pi^*(\tau^{2r-1})) = \{(W, v, x) \mid W \in \Sigma_r^G, v \in S^3 \text{ with } [v] = \pi(W), x \in (v^\perp)^{2r-1}\} / \sim$$

where $(v, x) \sim (\zeta v, \zeta x)$ for $\zeta \in S^1$.

We now explicitly define a map $\phi : E(\pi^*(\tau^{2r-1})) \rightarrow U_r^G$, which we later show is a G -equivariant homeomorphism. Let λ be a homomorphism with $r(\lambda) = r$ and let $X = [W, v, x] \in E(\pi^*(\tau^{2r-1}))$ with $W \in \Sigma_\lambda \subseteq \Sigma_r^G$. Write $x = (a_0 u, a_1 u, \dots, a_{2r-2} u)$ where $u \perp v$ and $a_j \in \mathbb{C}$. Since $\pi(W) = [v]$, the homomorphism $\lambda = \lambda(W)$ is given by $\lambda(z) = \begin{pmatrix} z^r & 0 \\ 0 & z^{-r} \end{pmatrix}$ written in the u, v basis. Let

$$e(z) = a_0 + a_1 z + \dots + a_{2r-2} z^{2r-2}.$$

Since $W \in \Sigma_\lambda$ we can write $W = W_{u,v}$ where

$$u = u(z) = \begin{pmatrix} z^r(1 + z^{-1}a(z^{-1})) \\ z^{-r-1}c(z^{-1}) \end{pmatrix}, \quad v = v(z) = \begin{pmatrix} z^{-r}b(z^{-1}) \\ z^{-r}(1 + z^{-1}d(z^{-1})) \end{pmatrix}$$

for some $a(w), b(z), c(w), d(w) \in \mathcal{O}$ and where the right hand sides are written with respect to the ordered basis u, v . We now explicitly define $\phi(X = [W, v, x]) \in U_r^G$ as follows. Let $P = AE$ where

$$A = \begin{pmatrix} 1 + z^{-1}a(z^{-1}) & b(z^{-1}) \\ z^{-2r-1}c(z^{-1}) & 1 + z^{-1}d(z^{-1}) \end{pmatrix}$$

for the a, b, c, d are the elements in \mathcal{O} above and

$$E = \begin{pmatrix} 1 & ze(z) \\ 0 & 1 \end{pmatrix}$$

(all written in the u, v basis). Define

$$(4.7) \quad V = \phi(W = W_{u,v}) := W_{P(z^r u), P(z^{-r} v)}.$$

Multiplying the matrices A and E shows that the subspace V thus defined is an element of U_r^G . Next we check that the construction of $\phi(W) = V$ given above is independent of the choices made. Suppose $X = [W, v', x']$ and suppose u' is orthogonal to v' . Then $u' = \zeta_1 u$, $v' = \zeta_2 v$, and $x' = \zeta_2 x$ for some $\zeta_1, \zeta_2 \in S^1$. In the construction given above we then obtain $e'(z) = \zeta_1^{-1} \zeta_2 e(z)$ instead of $e(z)$. In turn, E is replaced by

$$E' = \begin{pmatrix} 1 & \zeta_1^{-1} \zeta_2 z e(z) \\ 0 & 1 \end{pmatrix} = Z^{-1} E Z$$

where

$$Z = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}.$$

It is also straightforward to compute that the matrix A' which replaces A is $A' = Z^{-1} A Z$. Therefore P gets replaced by $P' := A' E' = Z^{-1} P Z$, and we obtain $V' = W_{P'(z^r u'), P'(z^{-r} v')} = W_{\zeta_1 P(z^r u), \zeta_2 P(z^{-r} v)}$, which is equal to V . Hence ϕ is well-defined.

The fact that ϕ is a bijection follows from solving equations to find A and E from P as in the proof of [22, Equation 8.4.4]. This gives a fibrewise inverse to ϕ . The map ϕ is also G -equivariant since by definition, the action of G on $\mathcal{K} = \mathcal{H} \otimes \mathbb{C}^2$ is via the standard action of G on the second factor. Hence

$$\phi(g \cdot X) = \phi([gW, gv, gx]) = W_{P(z^r(gu)), P(z^{-r}(gv))} = g \cdot V,$$

as desired. Finally, the topology on U_r^G is defined as a quotient of a subspace of $B(\mathcal{K})$, the bounded linear operators on \mathcal{K} , where two operators are equivalent if they define the same subspace. A map from a quotient space is continuous if and only if the composition with the quotient map is continuous, and the latter is given by matrix multiplications. Thus ϕ is continuous. The same argument applies to ϕ^{-1} . Hence ϕ is a G -equivariant homeomorphism. \square

The explicit description of U_r^G as a total space of a bundle in the previous theorem is a key tool that allows us to show our main theorem of this section (Theorem 4.20) that the inclusion of a certain subspace $S\text{Gr}'_{\text{bdd},r}$ (defined precisely in (4.8)) into $S\text{Gr}_{\alpha(\text{psm}),r}^z$ is a G -homotopy equivalence. However, we must first analyze more closely the relation between the spaces U_r^G and the spaces $S\text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K})$ discussed in previous sections. This requires a new description of the spaces U_λ and Σ_λ , which we will initially denote as \tilde{U}_λ and $\tilde{\Sigma}_\lambda$. (In Proposition 4.13 and Corollary 4.15 we show that in fact the two descriptions yield the same spaces.) Specifically, define

$$\begin{aligned} \tilde{U}_\lambda &:= \{W \in S\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) \mid \text{the orthogonal projection from } W \text{ to } \mathcal{K}_\lambda \text{ is an isomorphism}\}, \\ \tilde{U}_r^G &:= \cup_{|\lambda|=r} \tilde{U}_\lambda, \\ \tilde{\Sigma}_\lambda &:= \{W \in \tilde{U}_\lambda \mid r(W) = |\lambda|\}, \text{ and} \\ \tilde{\Sigma}_r^G &:= \cup_{|\lambda|=r} \tilde{\Sigma}_\lambda \end{aligned}$$

corresponding to the spaces Pressley-Segal denote as U_S and Σ_S in [22, pages 103 and 107].

Before proceeding we sketch the overall plan of the remainder of the (rather technical) argument leading to Theorem 4.20. First we prove that $\tilde{\Sigma}_r^G = \Sigma_r^G$ and $\tilde{U}_r^G = U_r^G$. We then use the new descriptions of the spaces Σ_r^G and U_r^G to show that $U_r^G \cap S\text{Gr}_{\alpha(\text{psm}),r-1}^z(\mathcal{K}) = U_r^G \setminus \Sigma_r^G$ and that $U_r^G \cup S\text{Gr}_{\alpha(\text{psm}),r-1}^z(\mathcal{K}) = S\text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. We also explicitly identify $U_r^G \setminus \Sigma_r^G$ with the complement of the zero cross-section of $\pi^*(\tau^{2r-1})$. Then, repeating the arguments thus far for the intersections of the relevant spaces with the subspaces $S\text{Gr}_{\text{bdd}}^z(\mathcal{K})$ of bounded weight, we obtain a description of $U_r^G \cap S\text{Gr}_{\text{bdd}}^z(\mathcal{K})$ as the total space of the pullback of τ^{2r-1} to $\Sigma_r^G \cap S\text{Gr}_{\text{bdd}}^z(\mathcal{K})$, with $(U_r^G \cap S\text{Gr}_{\alpha(\text{psm}),r-1}^z(\mathcal{K})) \cap S\text{Gr}_{\text{bdd}}^z(\mathcal{K})$ as the complement of the zero cross section. Finally, we use G -homotopy equivalences on the total spaces and complements of the zero cross-sections of the bundles (induced by a G -homotopy equivalence $\Sigma_r^G \cap S\text{Gr}_{\text{bdd}}^z(\mathcal{K}) \rightarrow \Sigma_r^G$ of the base spaces) as part of an induction argument to show that

$$S\text{Gr}'_{\text{bdd},r} := S\text{Gr}_{\text{bdd}}^z(\mathcal{K}) \cap S\text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K}) \hookrightarrow S\text{Gr}_{\alpha(\text{psm}),r}^z(\mathcal{K})$$

is a homotopy equivalence for each r .

With this broad outline in place, we proceed to the details of the argument.

Lemma 4.8. *Let U_λ and \tilde{U}_λ be as defined above. Then $U_\lambda \subset \tilde{U}_\lambda$.*

Proof. Let $W = W_{u,v}$. Orthogonal projection takes $u(z)$ to $z^r u_\lambda$ because it sends to 0 the multiples of the first basis element u_λ by z^k for $k < r$. It takes $v(z)$ to $z^{-r} v_\lambda$ since it sends to 0 the multiples of the second basis element v_λ by z^k for $k < -r$. Since both the domain and range of the projection is a free rank 2 module over $\mathbb{C}[z]$ and we have just shown that the map takes generators to generators, it is an isomorphism. \square

Lemma 4.9. *If $W \in \tilde{U}_r^G$ then $r(W) \leq r$.*

Proof. Suppose $W \in \tilde{U}_r^G$. Then $W \in \tilde{U}_\lambda$ for some λ with $|\lambda| = r$. If $x \in W$ with $\deg(x) < -r$, then the orthogonal projection from W to \mathcal{K}_λ takes x to 0, which is impossible since this projection is required to be an isomorphism. Thus W has no elements of degree $< -r$ and hence $r(W) \leq r$ as desired. \square

Lemma 4.10. *If $W \in \Sigma_r^G$ then $r(W) = r$. In particular, $\Sigma_r^G \subset \tilde{\Sigma}_r^G$.*

Proof. Suppose $W \in \Sigma_r^G$. By inspection, W contains an element of degree $-r$ (namely, the $v(z)$ from the definition), so $r(W) \geq r$. But $W \in \Sigma_r^G \subset U_r^G \subset \tilde{U}_r^G$, so $r(W) \leq r$. Thus $W \in \Sigma_r^G$ implies $r(W) = r$. \square

We also include two technical lemmas about holomorphic functions to be used in the proof of the proposition below.

Lemma 4.11. *Let $h(z) : S^1 \rightarrow \mathbb{C}$ be piecewise smooth. Suppose that the coefficient of z^k in the Fourier expansion of h is zero for $k < 0$. Then $h \in \mathcal{O}$.*

Proof. Let $\sum_{k=0}^{\infty} c_k z^k$ be the Fourier expansion of $h(z)$, where $c_k \in \mathbb{C}$. Since h is piecewise smooth, the Fourier expansion of $h(z)$ converges to $h(z)$. Since the series $\sum_{k=0}^{\infty} c_k z^k$ converges for all z with $|z| = 1$, its radius of convergence is greater than 1 so it defines a holomorphic function on the unit disk whose boundary value is $h(z)$. \square

Lemma 4.12. *Let $h(z)$ be holomorphic on a domain containing D^2 . Suppose that the restriction of $h(z)$ to S^1 is never 0 and that $h|_{S^1} : S^1 \rightarrow \mathbb{C} \setminus \{0\}$ is null homotopic. Then $h(z)$ has no zeros in D^2 .*

Proof. Consider the curve $\gamma(z) := h(S^1) \subset \mathbb{C}$. By hypothesis, γ is null homotopic. According to the Argument Principle

$$\# \text{ of zeros of } h(z) \text{ on } D^2 = \int_{S^1} \frac{h'(z)}{h(z)} dz = \int_{\gamma} \frac{1}{w} dw = \text{winding } \# \text{ of } \gamma \text{ about the origin} = 0.$$

Hence the origin is not in $h(D^2)$. \square

We are now in a position to prove the equivalence of our two definitions of Σ_r^G , corresponding to [22, Prop.8.4.1].

Proposition 4.13. *Let $\tilde{\Sigma}_r^G$ and Σ_r^G be as defined above. Then $\tilde{\Sigma}_r^G = \Sigma_r^G$.*

Proof. The containment $\Sigma_r^G \subset \tilde{\Sigma}_r^G$ is the content of Lemma 4.10. For the other containment, suppose $W \in \tilde{\Sigma}_r^G$. Then $W \in \tilde{\Sigma}_\lambda$ for some λ with $|\lambda| = r$. Let $x(z) \in W$ have degree $-r$. Then

$$x(z) = z^{-r} b(z^{-1}) u_\lambda + z^{-r} e(z^{-1}) v_\lambda$$

for $b(w), e(w) \in \mathcal{O}$. The orthogonal projection $W \rightarrow \mathcal{K}_\lambda$ (which is an isomorphism since $\tilde{\Sigma}_\lambda \subset \tilde{U}_\lambda$) takes $x(z)$ to $e_0 v_\lambda$, where e_0 is the constant term of $e(z^{-1})$. Hence $e_0 \neq 0$. Set $v(z) := x(z)/e_0$ and let $u(z)$ be the inverse image of $z^r u_\lambda$ under the projection $W \rightarrow \mathcal{K}_\lambda$. Since the orthogonal projection is an isomorphism, it follows that $W = W_{u,v}$ and this exhibits W as an element of Σ_λ as in (4.6), provided the holomorphicity and invertibility conditions are satisfied. Applying Lemma 4.11 to the components of $z^{-r} u(z^{-1})$ and $z^r v(z^{-1})$ shows that they are boundary values of holomorphic functions on $|z| > 1$. To see invertibility, let $A(z) \in \text{GL}(2)$ be the matrix whose columns are formed from $z^{-r} u(z)$ and $z^r v(z)$ and let $d(z) = \det A(z)$. According to Lemma 2.2, the homotopy class of the function $z \rightarrow d(z)$ is

$$-2 \text{Ind}(W_f) = 0 \in \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}.$$

Applying Lemma 4.12 to $d(z^{-1})$ shows that $d(z^{-1})$ is never zero on $|z| \geq 1$. Thus $W \in \Sigma_\lambda \subseteq \Sigma_r^G$. \square

Corollary 4.14. *The spaces $\{\Sigma_r^G\}$ form a stratification of $S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$ and $W \in \Sigma_r^G$ if and only if the rank of W is r .*

Proof. Suppose $W \in S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$. By Proposition 4.2 part (3), we know that $W \in \tilde{U}_{\lambda_W}$. On the other hand, by definition of the homomorphism λ_W (see (4.3)) we know $|\lambda_W| = r(W)$, so by definition of $\tilde{\Sigma}_\lambda^G$ we conclude $W \in \tilde{\Sigma}_{\lambda_W}^G$. By Proposition 4.13 this implies $W \in \Sigma_{r(W)}^G$. Since each element W of $S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$ has a unique rank we conclude the Σ_r^G form a stratification of $S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$, i.e.

$$S \text{Gr}_{\alpha(\text{psm})}^z(\mathcal{K}) = \coprod_r \Sigma_r^G.$$

□

We also get as a consequence the equivalence of the two definitions of U_r^G .

Corollary 4.15. *Let \tilde{U}_r^G and U_r^G be as defined above. Then $\tilde{U}_r^G = U_r^G$.*

Proof. The assertion that $U_r^G \subset \tilde{U}_r^G$ is the content of Lemma 4.8. Conversely, suppose $W \in \tilde{U}_\lambda$ with $|\lambda| = r$. The orthogonal projection $W \rightarrow \mathcal{K}_\lambda$ is an isomorphism, and therefore there exist unique $u(z), v(z) \in W$ projecting to $z^r u_\lambda$ and $z^{-r} v_\lambda$ respectively. Regarding W as an element of $\tilde{\Sigma}_{r(W)}^G = \Sigma_{r(W)}^G$ shows, as in the proof of Proposition 4.13, that $z^{-r} u(z^{-1})$ and $z^r v(z^{-1})$ are boundary values of holomorphic functions on a domain containing $|z| \geq 1$ and that the matrix whose columns are formed from these functions is invertible in $|z| \geq 1$. Thus the functions $u(z), v(z)$ exhibit W as an element of U_r^G . □

With the aid of our alternate descriptions of U_r^G and Σ_r^G , we can now relate $S \operatorname{Gr}_{\alpha(\text{psm})}^z(\mathcal{K})$ to our bundle description of U_r^G .

Proposition 4.16. *Under the G -equivariant identification $\phi : E(\pi^*(\tau^{2r-1})) \rightarrow U_r^G$, the intersection*

$$U_r^G \cap S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K})$$

is identified with the complement of the zero cross-section of $\pi^(\tau^{2r-1})$.*

Proof. Note that the inclusion $\Sigma_r^G \subset U_r^G$ corresponds, under the identification ϕ , with the inclusion of the zero cross-section into the total space. Suppose $W \in U_r^G \cap S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K})$. Since $r(W) < r$, Lemma 4.10 implies that $W \notin \Sigma_r^G$. That is, W lies in the complement of the zero cross-section of $\pi^*(\tau^{2r-1})$. Conversely, if $W \in U_r^G$ is not in Σ_r^G , by Lemma 4.10 its rank cannot be r and therefore it lies in $U_r^G \cap S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K})$. □

We also record the following, which again makes use of our alternate descriptions of U_r^G and Σ_r^G .

Proposition 4.17. *We have $S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K}) = U_r^G \cup S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K})$.*

Proof. Suppose $W \in U_r^G$. By Lemmas 4.8 and 4.9, $r(W) \leq r$ so $W \in S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K})$. Therefore $U_r^G \subset S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K})$, while $S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K}) \subset S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K})$ is trivial.

Conversely, suppose $W \in S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K})$. If $r(W) < r$ then $W \in S \operatorname{Gr}_{\alpha(\text{psm}), r-1}^z(\mathcal{K})$ while if $r(W) = r$ then $W \in \Sigma_r^G \subset U_r^G$. □

We now define the subset $S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K})$ referred to above, which is an important ingredient in our main theorem, as well as the bounded versions of the spaces U_r^G and Σ_r^G .

$$(4.8) \quad \begin{aligned} S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K}) &:= S \operatorname{Gr}_{\text{bdd}}^z(\mathcal{K}) \cap S \operatorname{Gr}_{\alpha(\text{psm}), r}^z(\mathcal{K}) \\ U_{\text{bdd}, r}^G(\mathcal{K}) &:= S \operatorname{Gr}_{\text{bdd}}^z(\mathcal{K}) \cap U_r^G \\ \Sigma_{\text{bdd}, r}^G(\mathcal{K}) &:= S \operatorname{Gr}_{\text{bdd}}^z(\mathcal{K}) \cap \Sigma_r^G \end{aligned}$$

Proposition 4.18. *We have $S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K}) \subset S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K})$ and $\bigcup_r S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K}) = \bigcup_r S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K})$.*

Proof. The inclusion $S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K}) \subset S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K})$ is a restatement of the fact that $W \in S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K})$ implies, according to Proposition 4.3, that $r(W) \leq r$. The containment

$$\bigcup_r S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K}) \subset \bigcup_r S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K})$$

follows. Conversely it is immediate from the definition that

$$\bigcup_r S \operatorname{Gr}'^z_{\text{bdd}, r}(\mathcal{K}) \subset S \operatorname{Gr}_{\text{bdd}}^z(\mathcal{K}) := \bigcup_r S \operatorname{Gr}_{\text{bdd}, r}^z(\mathcal{K}).$$

□

The bounded weight versions of the earlier results are recorded in Proposition 4.19. Since the arguments are the same as those given above (restricted to the appropriate subspaces), we omit the proofs.

Proposition 4.19. *Let $G = SU(2)$. Then:*

- (1) *The map $\pi : \Sigma_{\text{bdd},r}^G \rightarrow \mathbb{P}^1$ is a G -homotopy equivalence for all $r > 0$ with homotopy inverse s_r .*
- (2) *The space $U_{\text{bdd},0}^G = \Sigma_{\text{bdd},0}^G$ is G -equivariantly contractible.*
- (3) *For $r > 0$, $U_{\text{bdd},r}^G$ is G -homeomorphic to the total space of the bundle $\pi^*(\tau^{2r-1})$ over $\Sigma_{\text{bdd},r}^G$ (where π refers here to the restriction of π to $\Sigma_{\text{bdd},r}^G$).*
- (4) *Under the G -equivariant identification $\phi : E(\pi^*(\tau^{2r-1})) \rightarrow U_r^G$, the intersection*

$$U_{\text{bdd},r}^G \cap S\text{Gr}'^z_{\text{bdd},r-1}(\mathcal{K})$$

is identified with the complement of the zero cross-section of $\pi^(\tau^{2r-1})$ over $\Sigma_{\text{bdd},r}^G$.*

- (5) *The space $S\text{Gr}'^z_{\text{bdd},r}(\mathcal{K})$ is the union $U_{\text{bdd},r}^G \cup S\text{Gr}'^z_{\text{bdd},r-1}(\mathcal{K})$.*

We are now in a position to prove the main theorem of this section, Theorem 4.20. The basic idea is to make use of our homotopy equivalence $\Sigma_{\text{bdd},r}^G \simeq \Sigma_r^G$ on the base of our bundles to inductively show, applying a Mayer-Vietoris style argument, that $S\text{Gr}'^z_{\text{bdd},r} \rightarrow S\text{Gr}^z_{\alpha(\text{psm}),r}$ is a G -homotopy equivalence.

Theorem 4.20. *The inclusion $S\text{Gr}'^z_{\text{bdd},r} \rightarrow S\text{Gr}^z_{\alpha(\text{psm}),r}$ is a G -homotopy equivalence for all r .*

Proof. In general, if a topological G -space X is a union $U \cup V$, another G -space X' is also a union $U' \cup V'$, and $f : X \rightarrow X'$ is a map of G -spaces, assuming all of the inclusion maps are cofibrations, then f is a G -homotopy-equivalence if it induces G -homotopy-equivalences $U \rightarrow U'$, $V \rightarrow V'$ and $U \cap V \rightarrow U' \cap V'$. (See e.g. [24, Thm.7.1.8] for the non-equivariant version. Although [24] does not say so explicitly, all the maps constructed and used there are G -equivariant.) Thus our assertion follows by induction from the comparison of Prop. 4.17 with part (5) of Proposition 4.19, using the fact that the inclusion of the base space of a G -bundle into the associated total space is always a G -homotopy equivalence. \square

5. PROOF OF THE MAIN THEOREM

We are ready to prove the main result, Theorem 1.1. We do this by first showing that for $G = SU(2)$ the natural inclusion $\Omega_{\text{poly}}G \rightarrow \Omega G$ is an G -homotopy equivalence. This reduces the computation to that of $K_G^*(\Omega_{\text{poly}}G)$ and $K_T^*(\Omega_{\text{poly}}G)$, which was recorded in Theorem 3.6.

Our approach to the proof that $\Omega_{\text{poly}}G \simeq_G \Omega G$ is similar to that in [12], so we keep the explanation brief. We use G -equivariant versions of arguments given by Milnor in [19, Appendix A] to derive general conditions under which a map is an equivariant homotopy equivalence (it turns out to depend on the map restricting to equivariant homotopy equivalences on a sequence of subspaces, the union of which is the whole space). As is already pointed out in [12], although Milnor does not make explicit remarks concerning group actions, all the maps constructed and used in Milnor's proofs are equivariant.

Let H be a compact Lie group. Suppose $Z_0 \subset Z_1 \subset \dots \subset Z_n \subset \dots$ is an infinite sequence of spaces with H -action. Assume the inclusions $Z_i \hookrightarrow Z_{i+1}$ are H -equivariant and let $Z = \bigcup_{i=0}^{\infty} Z_i$ be their union. The **infinite mapping telescope** of Z (cf. [18]) is by definition the space

$$(5.1) \quad \begin{aligned} T_Z &:= Z_0 \times [0, 1] \cup Z_1 \times [1, 2] \cup \dots \cup Z_i \times [i, i+1] \cup \dots \\ &\subseteq Z \times \mathbb{R}. \end{aligned}$$

The H -action on $Z \times \mathbb{R}$ given by $g \cdot (z, t) = (gz, t)$ induces a H -action on the infinite mapping telescope.

Proposition 5.1. *Let Z_i , Z , and T_Z be as above. Assume that Z is paracompact. If for all $x \in Z$ there exists i such that x lies in the interior of Z_i , then the natural projection map $\pi_1 : T_Z \rightarrow Z$ is an H -homotopy equivalence.*

Proof. Since the group H is compact, we may use an H -invariant partition of unity to construct a map $f : Z \rightarrow [0, \infty)$ such that $f(x) \geq i + 1$ for $x \notin Z_i$. Then $g(x) := (x, f(x))$ is an H -equivariant homeomorphism from Z to $g(Z) \subset T_Z$, and the inclusion $j : g(Z) \hookrightarrow T_Z$ is an H -equivariant deformation retraction and satisfies $\pi_1 \circ j \circ g = 1_Z$. Therefore π_1 is an H -equivariant homotopy equivalence, as desired. \square

Theorem 5.2. *Let $Z = \bigcup_{i=0}^{\infty} Z_i$ and let $U = \bigcup_{i=0}^{\infty} U_i$. Assume that Z and U are paracompact. Let $f : Z \rightarrow U$ be a continuous H -equivariant map such that for each n , $f(Z_i) \subset U_i$ and the restriction $f_i := f|_{Z_i} : Z_i \rightarrow U_i$ is an H -homotopy equivalence. Then f is an H -homotopy equivalence.*

Proof. See [19, Appendix A]. All the maps in the cited reference are equivariant. \square

The preceding discussion, together with Theorem 4.20 and the fact that

$$\bigcup_r S \operatorname{Gr}_{\operatorname{bdd}, r}(\mathcal{K}) = \bigcup_r S \operatorname{Gr}'_{\operatorname{bdd}, r}(\mathcal{K}) = S \operatorname{Gr}_{\operatorname{bdd}}^z(\mathcal{K})$$

yields the following result.

Theorem 5.3. *The inclusion $S \operatorname{Gr}_{\operatorname{bdd}}(\mathcal{K}) \rightarrow S \operatorname{Gr}_{\operatorname{psm}}(\mathcal{K})$ is a G -homotopy equivalence. Equivalently, $\Omega_{\operatorname{poly}} SU(2) \rightarrow \Omega_{\operatorname{psm}} SU(2)$ is a G -homotopy equivalence.*

We also quote the following from [12].

Theorem 5.4.

Let $n \in \mathbb{Z}$ with $n > 0$. The natural inclusion $\Omega_{\operatorname{psm}} U(n) \hookrightarrow \Omega U(n)$ is an $SU(n)$ -equivariant homotopy equivalence.

Proof. The proof is again an application of Theorem 5.2, and is given in detail in [12]. \square

Using Theorem 3.6 together with Theorems 5.3 and 5.4) we can now describe the $R(G)$ -module and $R(T)$ -module structure of $K_G^*(\Omega G)$ and $K_T^*(\Omega G)$.

Theorem 5.5. *Let $G = SU(2)$ and let T denote its maximal torus. Let ΩG denote the space of based loops in G , equipped with the pointwise conjugation action of G . The $R(G)$ -module (respectively $R(T)$ -module) $K_G^*(\Omega G)$ (respectively $K_T^*(\Omega G)$) can be described as follows:*

$$\begin{aligned} K_G^q(\Omega G) &\cong K_G^q(\Omega_{\operatorname{poly}} G) \cong \varprojlim K_G^q(\Omega_{\operatorname{poly}, r} G) \cong \begin{cases} \prod_{r=0}^{\infty} R(G) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd;} \end{cases} \\ K_T^q(\Omega G) &\cong K_T^q(\Omega_{\operatorname{poly}} G) \cong \varprojlim K_T^q(\Omega_{\operatorname{poly}, r} G) \cong \begin{cases} \prod_{r=0}^{\infty} R(T) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases} \end{aligned}$$

Remark 5.6. *Note that our inverse limit becomes a direct product rather than a direct sum. The result, although a limit of free $R(G)$ -modules, is not itself a free $R(G)$ -module. (Recall from [5] that $\prod_{n=0}^{\infty} \mathbb{Z}$ is not a free abelian group.)*

Finally, we point out that our explicit computation implies in particular that, in this case, the W -invariants of $K_G^*(\Omega G)$ is precisely $K_T^*(\Omega G)$. (As we noted in the Introduction, this is not true of all G -spaces, cf. for instance [11, Example 4.8].)

Corollary 5.7. $K_G^*(\Omega G) = K_T^*(\Omega G)^W$.

Proof. Since $R(G) = R(T)^W$, this follows immediately from the right hand sides of the equalities given in Theorem 5.5. \square

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